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Gauge invariant Lagrangian construction for massive bosonic higher spin fields in D dimensions

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Abstract

We develop the BRST approach to Lagrangian formulation for massive higher integer spin fields on a flat space-time of arbitrary dimension. General procedure of gauge invariant Lagrangian construction describing the dynamics of massive bosonic field with any spin is given. No off-shell constraints on the fields (like tracelessness) and the gauge parameters are imposed. The procedure is based on construction of new representation for the closed algebra generated by the constraints defining an irreducible massive bosonic representation of the Poincare group. We also construct Lagrangian describing propagation of all massive bosonic fields simultaneously. As an example of the general procedure, we derive the Lagrangians for spin-1, spin-2 and spin-3 fields containing total set of auxiliary fields and gauge symmetries of free massive bosonic higher spin field theory.

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1 Introduction

Construction of higher spin field theory is one of the fundamental problems of high energy theoretical physics. At present, there exist the various approaches to this problem (see e.g. [1] for reviews and [2], [3] for recent development in massless and massive higher spin theories respectively) nevertheless the many aspects are still undeveloped.

The main problem of the higher spin field theory is introduction of interaction. One of the most important results there were a construction of consistent equations of motion for interacting higher spin fields [4] and finding the cubic interaction vertex of higher spin fields with gravity [5] in massless theory on a constant curvature space-time. The constructions of nonlinear equations of motion and cubic vertex were based on a specific gauge invariance of massless higher spin fields [4]. However problems of interacting massive higher spin fields have not been analysed so carefully as in massless case. Also we note that any string model contains an infinite number of massive string exitations. Therefore, the string models can be treat as a specific (infinite) collection of massive higher spin fields. Therefore, one can expect that interacting massive higher spin field theory should possesses some features of string theory.

The first Lagrangian description of the free massive fields with arbitrary spins in four dimensions was given in [6] where the problem of auxiliary fields was completely resolved. In this approach, the massive fields did not possess any gauge symmetry and satisfied the off-shell algebraic constraints like tracelessness for bosons or γ-tracelessness for fermions. Recently, the Lagrangian formulation of massive higher spin fields with some gauge symmetry (but still with the off-shell tracelessness constraints) was proposed in [7]. This approach was motivated by attempt to construct, at least approximate, an interaction of the massive higher spin fields with external electromagnetic and gravitational fields.

In this paper we study the massive higher spin fields on the base of BRST construction. Namely this techniques has been used in interacting open string field theory [8] (see also [9] for review). In some sense, the higher spin field theory is similar to the string field theory (see e.g. [10]) and one can hope the methods developed for string field theory will be successful in higher spin field theory as well. Attempts to construct an interacting massless higher spin theory analogously to string field theory have been undertaken in [11] where, in particular, the necessary and sufficient conditions for the existence of a gauge-invariant cubic interaction were found.

The BRST construction we use here arose at operator quntization of dynamical systems with first class constraints and, if to be more precise, it is called BRST-BFV construction or BFV construction [12] (see aslo the reviews [13])[1]. The systems under consideration are characterized by first class constraints in phase space T_a, $[T_a, T_b] = f_{ab}^c T_c$. Then BRST-BFV charge or BFV charge is constructed according to the rule

$$Q = \eta^a T_a + \frac{1}{2}\eta^b \eta^a f_{ab}^c \mathcal{P}_c, \qquad Q^2 = 0, \tag{1}$$

where η^a and $\mathcal{P}_a$ are canonically conjugate ghost variables (we consider here the case $gh(T) = 0$, then $gh(\eta^a) = 1$, $gh(\mathcal{P}_a) = -1$) satisfying the relations $\{\eta^a, \mathcal{P}_b\} = \delta^a_b$. After

[1] The BFV formalism we use differes from standard BRST formalism in configuration space of gauge theories [14]

quantization the BFV charge becomes an Hermitian operator acting in extended space of states including ghost operators, the physical states in the extended space are defined by the equation $Q|\Psi\rangle = 0$. Due to the nilpotency of the BRST-BFV operator, $Q^2 = 0$, the physical states are defined up to transformation $|\Psi'\rangle = |\Psi\rangle + Q|\Lambda\rangle$ which is treated as a gauge transformation. It is proved that there exists unitarizing Hamiltonian [12] leading to unitary S-matrix in subspace of physical states. Basic point here is classical Hamiltonian formulation of a Lagrangian model.

Application of BRST-BFV constructrion in the string field theory looks inverse to above quantization problem. The initial point is constraints in string theory, the BRST-BFV operator is constructed on the base of these constraints and finally an action, depending on string functional, is found on the base of BRST-BFV operator. We develop an analogous approach to massive higher spin field theory[2]. Generic strategy looks as follows. The constraints, defining the irreducible representation of the Poincare group with definite spin and mass (see e.g. [19]), are treated as the operators of first class constraints in extended space of states. However, as we will see, in the higher spin field theory the part of these constraints are non-Hermitian operators and in order to construct a Hermitian BRST operator we have to take into account the operators which are Hermitian conjugate to the initial constraints and which are not the constarints. Then for closing the algebra to the complete set of operators we must add some more operators which are not constraints as well[3]. Due to the presence of operators which are not the constraints the standard BRST construction can not be applied. One of the purpose of given paper is to show how to construct in this case a nilpotent operator analogous to BRST charge.

In this paper we discuss the gauge invariant Lagrangian description of the massive higher spin fields generalizing the BRST approach used for the massless fields [15, 16, 17, 18]. The method we use in the paper is based on further development of construction we formulated for massless fermionic higher spin fields [17]. As it will be shown this method can be applied to the massive theories as well and leads to gauge invariant theory. In contrast to all the previous works on massive higher spin gauge fields (see e.g. [6, 7]) we do not impose any off-shell constraints on the fields and the gauge parameters. All the constraints which define the irreducible massive higher spin representation will be consequences of equations of motion followed from the Lagrangian constructed and the gauge transformations.

The paper is organized as follows. In the next section we describe an algebra of operators generated by the constraints which are necessary to define an irreducible massive integer spin representation of Poincare group. It is shown that this algebra must include two operators which are not constraints. In order to be able to construct BRST operator and reproduce the equations of motion for higher spin fields we generalize in Section 3 the approach proposed in [17]. This generalization demands a construction of a new representation of the operator algebra having special structure. This new representation for the algebra under consideration is explicitly constructed in Section 4. Then in Section 5 we define the Lagrangian describing propagation of massive bosonic field of arbitrary fixed spin. There it is also shown that this theory is

[2]We follow further the notations generally accepted in string theory and BRST approach to massless higher spin fields and call BRST-BFV operator as BRST operator.

[3]This situation is illustrated in Section 3.

a gauge one and the gauge transformations are written down. Section 6 is devoted to construction of Lagrangian which describes propagation of all massive bosonic fields simultaneously. In Section 7 we illustrate the procedure of Lagrangian construction by finding the gauge invariant Lagrangians for massive spin-1, spin-2, and spin-3 fields and their gauge transformations in explicit form.

2 Algebra of operators generated by constraints

It is well known that the totally symmetric tensor field $\Phi_{\mu_1\cdots\mu_s}$, describing the irreducible spin-s massive representation of the Poincare group must satisfy the following constraints (see e.g. [19])

$$(\partial^2+m^2)\Phi_{\mu_1\cdots\mu_s}=0, \qquad \partial^{\mu_1}\Phi_{\mu_1\mu_2\cdots\mu_s}=0, \qquad \eta^{\mu_1\mu_2}\Phi_{\mu_1\cdots\mu_s}=0. \tag{2}$$

In order to describe all higher integer spin fields simultaneously it is convenient to introduce Fock space generated by creation and annihilation operators a_μ^+, a_μ with vector Lorentz index $\mu=0,1,2,\ldots,D-1$ satisfying the commutation relations

$$\left[a_\mu,a_\nu^+\right]=-\eta_{\mu\nu}, \qquad \eta_{\mu\nu}=(+,-,\ldots,-). \tag{3}$$

Then we define the operators

$$L_0=-p^2+m^2, \qquad L_1=a^\mu p_\mu, \qquad L_2=\tfrac{1}{2}a^\mu a_\mu, \tag{4}$$

where $p_\mu=-i\frac{\partial}{\partial x^\mu}$. These operators act on states in the Fock space

$$|\Phi\rangle \;=\; \sum_{s=0}^{\infty}\Phi_{\mu_1\cdots\mu_s}(x)a^{\mu_1+}\cdots a^{\mu_s+}|0\rangle \tag{5}$$

which describe all integer spin fields simultaneously if the following constraints on the states take place

$$L_0|\Phi\rangle=0, \qquad L_1|\Phi\rangle=0, \qquad L_2|\Phi\rangle=0. \tag{6}$$

If constraints (6) are fulfilled for the general state (5) then constraints (2) are fulfilled for each component $\Phi_{\mu_1\cdots\mu_s}(x)$ in (5) and hence the relations (6) describe all free massive higher spin bosonic fields simultaneously. Our purpose is to construct Lagrangian for the massive higher spin fields on the base of BRST approach, therefore first what we must construct is the Hermitian BRST operator. It means, we should have a system of Hermitian constraints. In the case under consideration the constraint L_0 is Hermitian, $L_0^+=L_0$, however the constraints L_1,L_2 are not Hermitian. We extend the set of the constraints L_0,L_1,L_2 adding two new operators $L_1^+=a^{\mu+}p_\mu$, $L_2^+=\frac{1}{2}a^{\mu+}a_\mu^+$. As a result, the set of operators L_0,L_1,L_2,L_1^+,L_2^+ is invariant under Hermitian conjugation.

Algebra of the operators L_0, L_1, L_1^+, L_2, L_2^+ is open in terms of commutators of these operators. We will suggest the following procedure of consideration. We want to use the BRST construction in the simplest (minimal) form coresponding to closed algebras. To get such an algebra we add to the above set of operators, all operators

	L_0	L_1	L_1^+	L_2	L_2^+	G_0	m^2
$L_0 = -p^2 + m^2$	0	0	0	0	0	0	0
$L_1 = p^\mu a_\mu$	0	0	$L_0 - m^2$	0	$-L_1^+$	L_1	0
$L_1^+ = p^\mu a_\mu^+$	0	$-L_0 + m^2$	0	L_1	0	$-L_1^+$	0
$L_2 = \frac{1}{2} a_\mu a^\mu$	0	0	$-L_1$	0	G_0	$2L_2$	0
$L_2^+ = \frac{1}{2} a_\mu^+ a^{\mu+}$	0	L_1^+	0	$-G_0$	0	$-2L_2^+$	0
$G_0 = -a_\mu^+ a^\mu + \frac{D}{2}$	0	$-L_1$	L_1^+	$-2L_2$	$2L_2^+$	0	0
m^2	0	0	0	0	0	0	0

Table 1: Operator algebra generated by the constraints

generated by the commutators of L_0, L_1, L_1^+, L_2, L_2^+. Doing such a way we obtain two new operators

$$m^2 \qquad \text{and} \qquad G_0 = -a_\mu^+ a^\mu + \frac{D}{2}. \tag{7}$$

The resulting algebra are written in Table 1. In this table the first arguments of the commutators and explicit expressions for all the operators are listed in the left column and the second argument of commutators are listed in the upper row. We will call this algebra as free massive integer higher spin symmetry algebra.

We emphisize that operators L_1^+, L_2^+ are not constraints on the space of ket-vectors. The constraints in space of ket-vectors are L_0, L_1, L_2 and they are the first class constraints in this space. Analogously, the constraints in space of bra-vectors are L_0, L_1^+, L_2^+ and they also are the first class constraints but only in this space, not in space of ket-vectors. Since the operator m^2 is obtained from the commutator

$$[L_1, L_1^+] = L_0 - m^2, \tag{8}$$

where L_1 is a constraint in the space of ket-vectors and L_1^+ is a constraint in the space of bra-vectors, then it can not be regarded as a constraint neither in the ket-vector space nor in the bra-vector space. It is easy to see that the operator m^2 is a central charge of the above algebra. Analogously the operator G_0 is obtained from the commutator

$$[L_2, L_2^+] = G_0, \tag{9}$$

where L_2 is a constraint in the space of ket-vectors and L_2^+ is a constraint in the space of bra-vectors. Therefore G_0 can not also be regarded as a constraint neither in the ket-vector space nor in the bra-vector space.

It is evident that naive construction of BRST operator for the system of operators given in Table 1 considering all of them as the first class constraints is contradictory and incorrect and as a consequence, such a construction can not reproduce the correct fundamental relations (6) (see e.g. [17] for the massless fermionic case). Further we suggest a new construction of nilponent operator corresponding to algebra given by Table 1 and reducing to standard BRST construction if all operators in closed algebra are constraints. Then this new construction is applied for derivation of the Lagrangian for massive higher spin fields. In order to clarify the basic features of the procedure let us consider a toy model, where we adapt BRST construction for operator algebras under consideration.

3 A toy model

Let us consider a model where the 'physical' states are defined by the equations

$$L_0|\Phi\rangle = 0, \qquad L_1|\Phi\rangle = 0, \tag{10}$$

with some operators L_0 and L_1. Let us also suppose that some scalar product $\langle\Phi_1|\Phi_2\rangle$ is defined for the states $|\Phi\rangle$ and let L_0 be a Hermitian operator $(L_0)^+ = L_0$ and let L_1 be non-Hermitian $(L_1)^+ = L_1^+$ with respect to this scalar product. In this section we show how to construct Lagrangian which will reproduce (10) as equations of motion up to gauge transformations.

In order to get the Lagrangian within BRST approach we should begin with the Hermitian BRST operator. However, the standard prescription does not allow to construct such a Hermitian operator on the base of operators L_0 and L_1 if L_1 is non-Hermitian. We assume to define the nilpotent Hermitian operator in the case under consideration as follows.

Let us consider the algebra generated by the operators L_0, L_1, L_1^+ and let this algebra takes the form

$$[L_0, L_1] = [L_0, L_1^+] = 0, \tag{11}$$
$$[L_1, L_1^+] = L_0 + C, \qquad C = const \neq 0. \tag{12}$$

In this algebra the central charge C plays the role analogous to m^2 and G_0 in the algebra given in Table 1. It is clear that the operator L_1^+ is not a constraint in sense of relations (10). We introduce the Hermitian BRST operator as if the operators L_0, L_1, L_1^+, C are the first class constraints

$$Q = \eta_0 L_0 + \eta_C C + \eta_1^+ L_1 + \eta_1 L_1^+ - \eta_1^+\eta_1(\mathcal{P}_0 + \mathcal{P}_C), \tag{13}$$
$$Q^2 = 0. \tag{14}$$

Here η_0, η_C, η_1, η_1^+ are the fermionic ghosts corresponding to the operators L_0, C, L_1^+, L_1 respectively, the $\mathcal{P}_0$, $\mathcal{P}_C$, $\mathcal{P}_1^+$, $\mathcal{P}_1$. are the momenta for the ghosts. These operators satisfy the usual commutation relations

$$\{\eta_0, \mathcal{P}_0\} = \{\eta_C, \mathcal{P}_C\} = \{\eta_1, \mathcal{P}_1^+\} = \{\eta_1^+, \mathcal{P}_1\} = 1 \tag{15}$$

and act on the vacuum state as follows

$$\mathcal{P}_0|0\rangle = \mathcal{P}_C|0\rangle = \eta_1|0\rangle = \mathcal{P}_1|0\rangle = 0. \tag{16}$$

The ghost numbers of these fields are

$$gh(\eta_0) = gh(\eta_C) = gh(\eta_1) = gh(\eta_1^+) = 1, \tag{17}$$
$$gh(\mathcal{P}_0) = gh(\mathcal{P}_C) = gh(\mathcal{P}_1) = gh(\mathcal{P}_1^+) = -1. \tag{18}$$

The operator Q (13) acts in the enlarge space on the state vectors depending also on the ghost fields η_0, η_C, η_1^+, $\mathcal{P}_1^+$

$$|\Psi\rangle \;=\; \sum_{k_i=0}^{1}(\eta_0)^{k_1}(\eta_C)^{k_2}(\eta_1^+)^{k_3}(\mathcal{P}_1^+)^{k_4}|\Phi_{k_1k_2k_3k_4}\rangle. \tag{19}$$

The states $|\Phi_{k_1k_2k_3k_4}\rangle$ in (19) do not depend on the ghosts and the state $|\Phi\rangle$ in (10) is a special case of (19) when $k_1 = k_2 = k_3 = k_4 = 0$.

Let us consider the equation

$$Q|\Psi\rangle = 0, \tag{20}$$

which defines the 'physical' states and which is treated as an equation of motion in BRST approach to higher spin field theory. It is natural to consider that the ghost number of the 'physical' states is zero and therefore we must leave in sum (19) only those terms which respect to this condition.

It is evident that if $|\Psi\rangle$ is a 'physical' state, then $|\Psi'\rangle = |\Psi\rangle + Q|\Lambda\rangle$ is also a 'physical' state for any $|\Lambda\rangle$ due to nilpotency of the BRST operator Q. Thus we have a gauge symmetry of equations of motion

$$\delta|\Psi\rangle = Q|\Lambda\rangle, \qquad\qquad gh(\Lambda) = -1. \tag{21}$$

Now we show that the approach when all the operators L_0, L_1, L_1^+, C are considered as the first class constraints in BRST (13) leads to contradictions with initial relations (10). For this purpose let us extract in the operator (13) and in the state (19) the dependence on the ghosts η_C, $\mathcal{P}_C$

$$Q \;=\; \eta_C C - \eta_1^+\eta_1\mathcal{P}_C + \Delta Q, \tag{22}$$
$$|\Psi\rangle \;=\; |\Psi_0\rangle + \eta_C|\Psi_1\rangle \tag{23}$$

and substitute them in the equation of motion (20) and the gauge transformation (21) (the part of gauge parameter $|\Lambda\rangle$ which depends on the ghost η_C is absent because in this term we can not respect its ghost number)

$$\Delta Q|\Psi_0\rangle - \eta_1^+\eta_1|\Psi_1\rangle = 0, \qquad\qquad \delta|\Psi_0\rangle = \Delta Q|\Lambda\rangle, \tag{24}$$
$$C|\Psi_0\rangle - \Delta Q|\Psi_1\rangle = 0, \qquad\qquad \delta|\Psi_1\rangle = C|\Lambda\rangle. \tag{25}$$

Now we gauge away $|\Psi_1\rangle$ and then we get a solution $|\Psi_0\rangle = 0$. But, $|\Psi_0\rangle = 0$ means $|\Phi\rangle = 0$ what contradicts to (10). This happens because we treat the operator C as a constraint. In order to get the correct result (10) we have to develope a new procedure.

We note that if we had $C=0$ in (12) and construct BRST operator as if L_0, L_1, L_1^+ were the first class constraints (it is clear that we do not introduce ghosts η_C, $\mathcal{P}_C$) we would reproduce equations of motion (10). Therefore, let us forget for a moment that L_1^+ is not a constraint and try to act as follows.

We enlarge the representation space of the operator algebra (11), (12) by introducing the additional (new) creation and annihilation operators and construct a new representation of the algebra bringing into it an arbitrary parameter h. The basic idea is to construct such a representation where the new operator C_{new} has the form $C_{new} = C + h$. Since parameter h is arbitrary and C is a central charge, we can choose $h = -C$ and the operator C_{new} will be zero in the new representation. After this we proceed as if operators L_{0new}, L_{1new}, L_{1new}^+ are the first class constraints.

Let us realize the above idea in explicit form for the toy model. We construct the new representation of the algebra (11), (12) so that the structure of the operators in this new representation be

$$\begin{array}{l}\text{New representaion}\\ \text{for an operator}\end{array} = \begin{array}{l}\text{Old representation}\\ \text{for the operator}\end{array} + \begin{array}{l}\text{Additional part, depending}\\ \text{on the additional creation and}\\ \text{annihilation operators}\\ \text{and parameter } h\end{array}. \qquad (26)$$

Since the additional creation and annihilation operators and the old ones commute with each other then we can construct a representation for the additional parts and add them to the initial expressions for the operators in algebra (11), (12)

$$L_{0new} = L_0 + L_{0add}, \qquad C_{new} = C + C_{add}, \qquad (27)$$

$$L_{1new} = L_1 + L_{1add}, \qquad L_{1new}^+ = L_1^+ + L_{1add}^+. \qquad (28)$$

The additional parts of the operators can be found if we demand algebra (11), (12) to have the same form in terms of new operators (27), (28). It is easy to check that a solution to additional parts can be written as follows

$$L_{0add} = 0, \qquad C_{add} = h, \qquad (29)$$

$$L_{1add} = hb, \qquad L_{1add}^+ = b^+. \qquad (30)$$

Here we have introduced the new bosonic creation and annihilation operators b^+, b with the standard commutation relations

$$[b, b^+] = 1. \qquad (31)$$

Now we substitute (29), (30) into (27), (28) and find the new representation for the algebra (11), (12)

$$L_{0new} = L_0, \qquad C_{new} = C + h, \qquad (32)$$

$$L_{1new} = L_1 + hb, \qquad L_{1new}^+ = L_1^+ + b^+. \qquad (33)$$

Thus we have constructed the new representation. In principle, we could set $h = -C$ and get $C_{new} = 0$, but we will follow another equivalent scheme. Namley we still consider C_{new} as nonzero operator including the arbitrary parameter h, but demand for state vectors and gauge parameters to be independent on ghost η_C as before. We will see that these conditions reproduce that h should be equal to $-C$.

We introduce the BRST construction taking the operators in new representation as if they were the first class constarints. It leads to

$$Q_h = \eta_0 L_0 + \eta_C C_{new} + \eta_1^+ L_{1new} + \eta_1 L_{1new}^+ - \eta_1^+ \eta_1 (\mathcal{P}_0 + \mathcal{P}_C), \tag{34}$$
$$Q_h^2 = 0. \tag{35}$$

These new operators (32), (33) together with BRST operator (34) act on the states of the enlarged space which are independent on ghost η_C (according to the scheme described above) but include the new operators b^+

$$|\Psi\rangle = \sum_{k=0}^{\infty} \sum_{k_i=0}^{1} (\eta_0)^{k_1} (\eta_1^+)^{k_2} (\mathcal{P}_1^+)^{k_3} (b^+)^k |\Phi_{kk_1k_2k_3}\rangle. \tag{36}$$

Let us show that eq. (20) with BRST operator (34) and the state vector (36) have solution (10) up to gauge transformations, that is the above scheme indeed leads us to the desirable relations (10).

For this purpose let us extract in the operator (34) the dependence on the ghosts η_C, $\mathcal{P}_C$

$$Q_h = \eta_C (C + h) - \eta_1^+ \eta_1 \mathcal{P}_C + \Delta Q_h. \tag{37}$$

Then equation (20) and gauge transformation (21) yield

$$\Delta Q_h |\Psi\rangle = 0, \qquad \delta |\Psi\rangle = Q_h |\Lambda\rangle, \tag{38}$$
$$(C + h)|\Psi\rangle = 0, \qquad (C + h)|\Lambda\rangle = 0. \tag{39}$$

From (39) we find that parameter $h = -C$. Then we extract the dependence of the state vector and the gauge parameter on the ghost fields

$$|\Psi\rangle = |\Psi_0\rangle + \eta_1^+ \mathcal{P}_1^+ |\Psi_1\rangle + \eta_0 \mathcal{P}_1^+ |\Psi_2\rangle, \qquad |\Lambda\rangle = \mathcal{P}_1^+ |\lambda\rangle. \tag{40}$$

Here the vectors $|\Psi_0\rangle$, $|\Psi_1\rangle$, $|\Psi_2\rangle$, $|\lambda\rangle$ are independent of the ghost fields and depend on operator b^+

$$|\Psi_A\rangle = \sum_{k=0}^{\infty} (b^+)^k |0\rangle \otimes |\Phi_{Ak}\rangle, \qquad A = 0, 1, 2 \tag{41}$$
$$|\lambda\rangle = \sum_{k=0}^{\infty} (b^+)^k |0\rangle \otimes |\lambda_k\rangle, \tag{42}$$

with $|0\rangle$ being the vacuum for the operator b: $b|0\rangle = 0$. The state $|\Phi\rangle$ which stands in (10) is $|\Phi_{00}\rangle$ in notations of (41).

Now ones write down the equations of motion

$$L_0 |\Psi_0\rangle - (L_1^+ + b^+)|\Psi_2\rangle = 0, \tag{43}$$
$$(L_1 - Cb)|\Psi_0\rangle - (L_1^+ + b^+)|\Psi_1\rangle - |\Psi_2\rangle = 0, \tag{44}$$
$$L_0 |\Psi_1\rangle - (L_1 - Cb)|\Psi_2\rangle = 0 \tag{45}$$

and the gauge transformations

$$\delta|\Psi_0\rangle = (L_1^+ + b^+)|\lambda\rangle, \qquad \delta|\Psi_1\rangle = (L_1 - Cb)|\lambda\rangle, \qquad \delta|\Psi_2\rangle = L_0|\lambda\rangle. \quad (46)$$

Now with the help of the gauge transformations we can remove the field $|\Psi_2\rangle$, after this we have the gauge transformation with the constrained gauge parameter $|\lambda\rangle$

$$L_0|\lambda\rangle = 0. \quad (47)$$

Since one of the equations of motion becomes

$$L_0|\Psi_1\rangle = 0 \implies L_0|\Phi_{1k}\rangle = 0, \qquad \text{for all } k \quad (48)$$

we can remove the field $|\Psi_1\rangle$

$$\delta|\Phi_{1k}\rangle = L_1|\lambda_k\rangle - (k+1)C|\lambda_{k+1}\rangle \quad (49)$$

using this constrained gauge parameter (47). After this we have the constrained gauge parameter (47) which does not depend on b^+: $|\lambda\rangle = |0\rangle \otimes |\lambda_0\rangle$. We use it to remove the component of $|\Psi_0\rangle$ which is linear in b^+ $(b^+|0\rangle \otimes |\Phi_{01}\rangle)$

$$\delta|\Phi_{01}\rangle = |\lambda_0\rangle. \quad (50)$$

Now the components of $|\Psi_0\rangle$ which depend on b^+ $((b^+)^k|0\rangle \otimes |\Phi_{0k}\rangle,\ k \geq 2)$ vanish as consequence of equation of motion (44). It remains only $|\Psi_0\rangle$ which is independent of b^+ $(|0\rangle \otimes |\Phi_{00}\rangle)$ and equations of motion for $|\Phi_{00}\rangle$ which follow from (43), (44)

$$L_0|\Psi_0\rangle = 0 \implies L_0|\Phi_{00}\rangle = 0, \quad (51)$$

$$(L_1 - Cb)|\Psi_0\rangle = 0 \implies L_1|\Phi_{00}\rangle = 0 \quad (52)$$

coincide with (10). Thus we have shown that the scheme described above leads us to the desirable result (10). There are no any contradictions, the procedure works perfectly.

Also we have shown that the presence of operators which are Hermitian conjugate to constraints like L_1^+ does not lead to new restrictions on the physical state. This is explained by the fact that L_1^+ appears in the BRST operator being multiplied with ghost annihilation operators η_1 (16) which kill the 'physical' states $|0\rangle \otimes |\Phi_{00}\rangle$ in (41). The presence of operators like L_1^+ in BRST operator enlarge the gauge symmetry of a theory only.

Now we want to say once again that there are two equivalent ways of constructing BRST operator. First of them consists in putting $h = -C$ in all the formulas for the new expressions for the operators (moreover we can not introduce this parameter at all and construct new representation for the algebra so that $C_{new} = 0$) and then construct BRST operator without ghosts η_C, $\mathcal{P}_C$. Another way consists in leaving the parameter h arbitrary and constructing BRST operator with ghosts η_C, $\mathcal{P}_C$ in order to define this parameter h later as a consequence of equation of motion (20). Both of these ways will be used in constructing the new representation for the operators of the algebra given in Table 1. The first one will be used for the operator m^2 and the second one will be used for the operator G_0.

We pay attention that operators L_{1new} and L^+_{1new} are not mutually conjugate in the new representation if we use the usual rules for Hermitian conjugation of the additional creation and annihilation operators

$$(b)^+ = b^+, \qquad (b^+)^+ = b. \tag{53}$$

To consider the operators L_{1new}, L^+_{1new} as conjugate to each other we change a definition of scalar product for the state vectors (36) as follows

$$\langle\Psi_1|\Psi_2\rangle_{new} = \langle\Psi_1|K_h|\Psi_2\rangle, \tag{54}$$

with

$$K_h = \sum_{n=0}^{\infty} |n\rangle \frac{h^n}{n!}\langle n|, \tag{55}$$

$$|n\rangle = (b^+)^n|0\rangle. \tag{56}$$

Now the new operators L_{1new}, L^+_{1new} are mutually conjugate and the operator Q_h is Hermitian relatively the new scalar product (54) since the following relations take place

$$K_h L^+_{1new} = (L_{1new})^+ K_h, \quad K_h L_{1new} = (L^+_{1new})^+ K_h, \quad Q^+_h K_h = K_h Q_h. \tag{57}$$

Finally we note that equations of motion (43)–(45) may be derived from the following Lagrangian

$$\mathcal{L} = \int d\eta_0 \langle\Psi|K_{-C}\Delta Q_{-C}|\Psi\rangle \tag{58}$$

where subscripts $-C$ means that we substitute $-C$ instead of h. Here the integral is taken over Grassmann odd variable η_0.

In the next sections we describe application of this procedure in case of the operator algebra given in Table 1.

4 New representation for the algebra

The analysis of the toy model in the previous section teaches us how to develop a BRST approach in the case when operator algebra given by Table 1 contains the Hermitian operators m^2 and G_0 which are not constraints neither in the space of ket-vectors nor in the space of bra-vectors. Naive use of these operators in BRST construction yields to contradictions with the basic relations (6). According to analysis carried out in Section 3, in order to avoid the contradictions we should construct a new representation of the algebra with two arbitrary parameters h_m and h for new operators m^2_{new} and G_{0new} respectively. Then we choose one of them h_m so that $m^2_{new} = 0$ and do not introduce the corresponding ghosts η_m, $\mathcal{P}_m$ in the BRST operator, but the second one h we leave arbitrary. Besides we know from Section 3 that the presence of operators which are Hermitian conjugate to constrains does not lead to any contradictions in the approach under consideration.

The purpose of this Section is to construct a new representation for the algebra of the operators given in Table 1 assuming the new expressions for the operators in the form analogous to (26)

$$L_{0new} = L_0 + L_{0add}, \qquad G_{0new} = G_0 + G_{0add}, \qquad m^2_{new} = m^2 + m^2_{add} = 0, \tag{59}$$
$$L_{1new} = L_1 + L_{1add}, \qquad L^+_{1new} = L^+_1 + L^+_{1add}, \tag{60}$$
$$L_{2new} = L_2 + L_{2add}, \qquad L^+_{2new} = L^+_2 + L^+_{2add}, \tag{61}$$

where the additional parts should depend only on the new creation and annihilation operators (and possibly on h, h_m). Besides, the new operator G_0 must be linear in the parametrs h. It means

$$G_{0add} = \Delta G_{0add} + h, \tag{62}$$

where ΔG_{0add} depends only on additional creation and annihilation operators.

The basic principle for finding the new representaion is preservation of the operator algebra given in Table 1. Since the initial operators commute with the additional parts, we have to construct a representation only for these additional parts. To do that, we introduce two pairs of additional bosonic annihilation and creation operators b_1, b^+_1, b_2, b^+_2 so that the complete operators (59)–(61) satisfy the initial algebra. One can check that a proper solution to the additional parts can be written as follows

$$m^2_{add} = -m^2, \qquad G_{0add} = b^+_1 b_1 + \tfrac{1}{2} + 2b^+_2 b_2 + h, \tag{63}$$
$$L_{0add} = 0, \tag{64}$$
$$L^+_{1add} = m b^+_1, \qquad L_{1add} = m b_1, \tag{65}$$
$$L^+_{2add} = -\tfrac{1}{2} b^{+2}_1 + b^+_2, \qquad L_{2add} = -\tfrac{1}{2} b^2_1 + (b^+_2 b_2 + h) b_2. \tag{66}$$

The operators b^+_1, b_1, b^+_2, b_2 satisfy the standard commutation relations

$$[\, b_1, b^+_1] = [\, b_2, b^+_2] = 1. \tag{67}$$

Thus we have the new representation of the operator algebra. It is given by (59)–(61) with the additional parts (63)–(66) found in explicit form.

It is easy to see, the operators (66) are not Hermitian conjugate to each other

$$(L_{2add})^+ \neq L^+_{2add} \tag{68}$$

if we use the usual rules for Hermitian conjugation of the additional creation and annihilation operators relatively the standard scalar product in Fock space

$$(b_1)^+ = b^+_1, \qquad (b_2)^+ = b^+_2. \tag{69}$$

Like in Section 3 we change the definition of scalar product of vectors in the new representation as follows

$$\langle \Phi_1 | \Phi_2 \rangle_{new} = \langle \Phi_1 | K | \Phi_2 \rangle, \tag{70}$$

with some operator K. This operator K can be found in the form

$$K = \sum_{n=0}^{\infty} |n\rangle \frac{C(n,h)}{n!} \langle n|, \tag{71}$$

$$|n\rangle = (b_2^+)^n |0\rangle, \tag{72}$$

$$C(n,h) = h(h+1)(h+2)\ldots(h+n-1), \qquad C(0,h) = 1. \tag{73}$$

Using the equations (71), (73) one can show that the following relations take place

$$KL_{2new} = (L_{2new}^+)^+ K, \qquad KL_{2new}^+ = (L_{2new})^+ K. \tag{74}$$

It means that the operators L_{2new}, L_{2new}^+ are conjugate to each other relatively the scalar product (70) with operator K given by (71).

Now we introduce the operator $\tilde{Q}$ on the base of new operators using BRST construction as if all these operators were the first class constarints. As a result ones get

$$\begin{aligned}
\tilde{Q} &= \eta_0 L_0 + \eta_1^+ L_{1new} + \eta_1 L_{1new}^+ + \eta_2^+ L_{2new} + \eta_2 L_{2new}^+ + \eta_G G_{0new} \\
&\quad - \eta_1^+ \eta_1 \mathcal{P}_0 - \eta_2^+ \eta_2 \mathcal{P}_G + (\eta_G \eta_1^+ + \eta_2^+ \eta_1) \mathcal{P}_1 + (\eta_1 \eta_G + \eta_1^+ \eta_2) \mathcal{P}_1^+ \\
&\qquad + 2\eta_G \eta_2^+ \mathcal{P}_2 + 2\eta_2 \eta_G \mathcal{P}_2^+,
\end{aligned} \tag{75}$$

$$\tilde{Q}^2 = 0. \tag{76}$$

Here η_0, η_1^+, η_1, η_2^+, η_2, η_G are the fermionic ghosts corresponding to the operators L_0, L_{1new}, L_{1new}^+, L_{2new}, L_{2new}^+, G_{0new} respectively. The momenta for these ghosts are $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_1^+$, $\mathcal{P}_2$, $\mathcal{P}_2^+$, $\mathcal{P}_G$. The ghost operators satisfy the usual commutation relations

$$\{\eta_0, \mathcal{P}_0\} = \{\eta_G, \mathcal{P}_G\} = \{\eta_1, \mathcal{P}_1^+\} = \{\eta_1^+, \mathcal{P}_1\} = \{\eta_2, \mathcal{P}_2^+\} = \{\eta_2^+, \mathcal{P}_2\} = 1, \tag{77}$$

and act on the vacuum state as follows

$$\mathcal{P}_0|0\rangle = \mathcal{P}_G|0\rangle = \eta_1|0\rangle = \mathcal{P}_1|0\rangle = \eta_2|0\rangle = \mathcal{P}_2|0\rangle = 0. \tag{78}$$

We assume that the introduced operator (75) acts in the enlarged space of state vectors depending on $a^{+\mu}$, b_1^+, b_2^+ and on the ghost operators η_0, η_1^+, $\mathcal{P}_1^+$, η_2^+, $\mathcal{P}_2^+$. Ones emphasize that the state vectors must be independent of the ghost η_G corresponding to the operator G_0. The general structure of such a state is

$$\begin{aligned}
|\chi\rangle &= \sum_{k_i} (b_1^+)^{k_1} (b_2^+)^{k_2} (\eta_0)^{k_3} (\eta_1^+)^{k_4} (\mathcal{P}_1^+)^{k_5} (\eta_2^+)^{k_6} (\mathcal{P}_2^+)^{k_7} \times \\
&\qquad \times a^{+\mu_1} \cdots a^{+\mu_{k_0}} \chi^{k_1 \cdots k_7}_{\mu_1 \cdots \mu_{k_0}}(x)|0\rangle.
\end{aligned} \tag{79}$$

The sum in (79) is taken over k_0, k_1, k_2, running from 0 to infinity and over k_3, k_4, k_5, k_6, k_7 running from 0 to 1. Besides for the 'physical' states we must leave in the sum (79) only those terms which ghost number is zero. It is evident that the state vectors (5) are the partial cases of the above vectors.

One can show that the operator (75) satisfy the relation

$$\tilde{Q}^{+}K \quad = \quad K\tilde{Q}. \tag{80}$$

It means this operator is Hermitian relatively the scalar product (70) with operator K (71).

Now we turn to the construction of the Lagrangians for free massive bosonic higher spin fields.

5 Lagrangians for the massive bosonic field with given spin

In this Section we construct Lagrangians for free massive bosonic higher spin gauge fields using the BRST operator (75).

First, ones extract the dependence of the BRST operator (75) on the ghosts η_G, $\mathcal{P}_G$

$$\tilde{Q} \quad = \quad Q + \eta_G(\sigma + h) - \eta_2^{+}\eta_2\mathcal{P}_G, \tag{81}$$

$$Q^2 = \eta_2^{+}\eta_2(\sigma + h), \qquad [Q, \sigma] = 0, \tag{82}$$

with

$$\sigma \quad = \quad G_0 + b_1^{+}b_1 + \tfrac{1}{2} + 2b_2^{+}b_2 + \eta_1^{+}\mathcal{P}_1 - \eta_1\mathcal{P}_1^{+} + 2\eta_2^{+}\mathcal{P}_2 - 2\eta_2\mathcal{P}_2^{+}, \tag{83}$$

$$\begin{aligned} Q \quad = \quad & \eta_0 L_0 + \eta_1^{+}L_{1new} + \eta_1 L_{1new}^{+} + \eta_2^{+}L_{2new} + \eta_2 L_{2new}^{+} \\ & - \eta_1^{+}\eta_1\mathcal{P}_0 + \eta_2^{+}\eta_1\mathcal{P}_1 + \eta_1^{+}\eta_2\mathcal{P}_1^{+}. \end{aligned} \tag{84}$$

After this, the equation on the 'physical' states (79) in the BRST approach $\tilde{Q}|\chi\rangle = 0$ yields two equations

$$Q|\chi\rangle = 0, \tag{85}$$

$$(\sigma + h)|\chi\rangle = 0. \tag{86}$$

From equation (86) we find the possible values of h. The equation (86) is the eigenvalue equation for the operator σ (83) with the corresponding eigenvalues $-h$

$$-h \quad = \quad n + \frac{D-5}{2}, \qquad n = 0, 1, 2, \dots \ . \tag{87}$$

Let us denote the eigenvectors of the operator σ corresponding to the eigenvalues $n + \frac{D-5}{2}$ as $|\chi\rangle_n$

$$\sigma|\chi\rangle_n \quad = \quad \left(n + \frac{D-5}{2}\right)|\chi\rangle_n. \tag{88}$$

Since

$$|\chi\rangle_n \quad = \quad a^{\mu_1+}\cdots a^{\mu_n+}\Phi_{\mu_1\cdots\mu_n}(x)|0\rangle + \dots, \tag{89}$$

where the dots denote terms depending on the ghosts fields and/or on the operators b_1^+, b_2^+ (these fields are auxiliary ones for the physical field $\Phi_{\mu_1\cdots\mu_n}(x)$ in (89)), then the numbers n are related with the spin s of the corresponding eigenvectors as $s = n$.

The solutions to the system of equations (85), (86) are enumerated by $n = 0, 1, 2, \ldots$ and satisfy the equations

$$Q_n|\chi\rangle_n \;=\; 0, \tag{90}$$

where in the BRST operator (84) we substituted $n + \frac{D-5}{2}$ instead of $-h$. Thus we get the BRST operator depending on n

$$\begin{aligned} Q_n \;=\;& \eta_0 L_0 + \eta_1^+ L_{1new} + \eta_1 L_{1new}^+ + \eta_2^+\Big(L_2 - \tfrac{1}{2}b_1^2 + b_2^+ b_2^2\Big) + \eta_2 L_{2new}^+ \\ & - \eta_1^+\eta_1\mathcal{P}_0 + \eta_2^+\eta_1\mathcal{P}_1 + \eta_1^+\eta_2\mathcal{P}_1^+ - \eta_2^+ b_2\big(n + \tfrac{D-5}{2}\big). \end{aligned} \tag{91}$$

Then we rewrite the operators Q_n (91) in the form independent of n. This is done by replacing $n + \frac{D-5}{2}$ in (91) by the operator σ (83). It leads to

$$\begin{aligned} Q_\sigma \;=\;& \eta_0 L_{0new} + \eta_1^+ L_{1new} + \eta_1 L_{1new}^+ + \eta_2^+\Big(L_2 - \tfrac{1}{2}b_1^2 + b_2^+ b_2^2\Big) + \eta_2 L_{2new}^+ \\ & - \eta_1^+\eta_1\mathcal{P}_0 + \eta_2^+\eta_1\mathcal{P}_1 + \eta_1^+\eta_2\mathcal{P}_1^+ - \eta_2^+ b_2\sigma, \end{aligned} \tag{92}$$

where $Q_\sigma = Q_n|_{n+\frac{D-5}{2}\to\sigma}$. One can check that the operator Q_σ is nilpotent.

Now we turn to the gauge transformations. We suppose that the parameters of the gauge transformations are also independent of η_G. Due to eq. (82) we have the following gauge transformations and the corresponding eigenvalue equations for the gauge parameters

$$\delta|\chi\rangle = Q|\Lambda\rangle, \qquad (\sigma + h)|\Lambda\rangle = 0, \qquad gh(|\Lambda\rangle) = -1, \tag{93}$$

$$\delta|\Lambda\rangle = Q|\Omega\rangle, \qquad (\sigma + h)|\Omega\rangle = 0, \qquad gh(|\Omega\rangle) = -2, \tag{94}$$

where h has been determined (87).

Next step is to extract the Hermitian ghost mode from the operator Q_σ (92). This operator has the structure

$$Q_\sigma = \eta_0 L_0 - \eta_1^+\eta_1\mathcal{P}_0 + \Delta Q_\sigma, \tag{95}$$

where ΔQ_σ is independent of η_0, $\mathcal{P}_0$

$$\begin{aligned} \Delta Q_\sigma \;=\;& \eta_1^+ L_{1new} + \eta_1 L_{1new}^+ + \eta_2^+\Big(L_2 - \tfrac{1}{2}b_1^2 + b_2^+ b_2^2\Big) + \eta_2 L_{2new}^+ \\ & + \eta_2^+\eta_1\mathcal{P}_1 + \eta_1^+\eta_2\mathcal{P}_1^+ - \eta_2^+ b_2\sigma. \end{aligned} \tag{96}$$

We decompose the state vector of a given spin $s = n$ as (88)

$$|\chi\rangle_n = |S\rangle_n + \eta_0|A\rangle_n. \tag{97}$$

and find the equations of motion which follow from (90)

$$\Delta Q_\sigma|S\rangle_n - \eta_1^+\eta_1|A\rangle_n = 0, \tag{98}$$

$$L_0|S\rangle_n - \Delta Q_\sigma|A\rangle_n = 0. \tag{99}$$

Ones may check that these equations can be derived from the following Lagrangian[4]

$$\begin{aligned}\mathcal{L}_n &= {}_n\langle S|K_n L_0|S\rangle_n - {}_n\langle S|K_n \Delta Q_\sigma|A\rangle_n \\ &\qquad - {}_n\langle A|K_n \Delta Q_\sigma|S\rangle_n + {}_n\langle A|K_n \eta_1^+\eta_1|A\rangle_n, \end{aligned} \tag{100}$$

which can also be written in more concise form as

$$\mathcal{L}_n = \int d\eta_0 \; {}_n\langle \chi|K_n Q_\sigma|\chi\rangle_n \tag{101}$$

with $|\chi\rangle_n$ (97), Q_σ (92) and K_n is the operator (71) where the substitution $-h \to n + \frac{D-5}{2}$ is assumed. The integral in (101) is taken over Grassmann odd variable η_0.

Now we turn to the symmetry transformations which follow from (93), (94). After the decomposition of the gauge parameters on η_0

$$|\Lambda\rangle = |\Lambda_0\rangle + \eta_0|\Lambda_1\rangle, \tag{102}$$

$$|\Omega\rangle = |\Omega_0\rangle \tag{103}$$

(the part of $|\Omega\rangle$ which depends on η_0 is absent because in this term we can't respect its ghost number) we find the symmetry transformations for the fields

$$\delta|S\rangle_n = \Delta Q_\sigma|\Lambda_0\rangle_n - \eta_1^+\eta_1|\Lambda_1\rangle_n, \tag{104}$$

$$\delta|A\rangle_n = L_0|\Lambda_0\rangle_n - \Delta Q_\sigma|\Lambda_1\rangle_n, \qquad gh(|\Lambda_i\rangle_n) = -(i+1) \tag{105}$$

and symmetry transformations for the gauge parameters

$$\delta|\Lambda_0\rangle_n = \Delta Q_\sigma|\Omega_0\rangle_n, \qquad \delta|\Lambda_1\rangle_n = L_0|\Omega_0\rangle_n, \qquad gh(|\Omega_0\rangle_n) = -2. \tag{106}$$

Let us show that the Lagrangian (100) describes a bosonic massive higher spin field. First we get rid of the gauge parameter $|\Lambda_1\rangle_n$ and then we get rid of the field $|A\rangle_n$ using their symmetry transformations. Thus ones get the field $|S\rangle_n$ and the constrained gauge parameter $|\Lambda_0\rangle_n$ ($L_0|\Lambda_0\rangle_n = 0$). Further we will omit the subscript n at the state vectors and the gauge parameters. Next decomposing the state vector $|S\rangle$ and the gauge parameter $|\Lambda_0\rangle$ on the ghost fields

$$\begin{aligned}|S\rangle &= |S_1\rangle + \eta_1^+\mathcal{P}_1^+|S_2\rangle + \eta_1^+\mathcal{P}_2^+|S_3\rangle \\ &\qquad + \eta_2^+\mathcal{P}_1^+|S_4\rangle + \eta_2^+\mathcal{P}_2^+|S_5\rangle + \eta_1^+\eta_2^+\mathcal{P}_1^+\mathcal{P}_2^+|S_6\rangle, \end{aligned} \tag{107}$$

$$|\Lambda_0\rangle = \mathcal{P}_1^+|\lambda_1\rangle + \mathcal{P}_2^+|\lambda_2\rangle + \eta_1^+\mathcal{P}_1^+\mathcal{P}_2^+|\lambda_3\rangle + \eta_2^+\mathcal{P}_1^+\mathcal{P}_2^+|\lambda_4\rangle \tag{108}$$

and substituting into (98), (99), (104) ones get the equations of motion

$$L_0|S_1\rangle = L_0|S_2\rangle = L_0|S_3\rangle = L_0|S_4\rangle = L_0|S_5\rangle = L_0|S_6\rangle = 0, \tag{109}$$

$$L_{1new}|S_1\rangle - L_{1new}^+|S_2\rangle - L_{2new}^+|S_3\rangle = 0, \tag{110}$$

$$L_2'|S_1\rangle + |S_2\rangle - L_{2new}^+|S_5\rangle - L_{1new}^+|S_4\rangle = 0, \tag{111}$$

$$L_2'|S_2\rangle + |S_5\rangle - L_{1new}|S_4\rangle + L_{2new}^+|S_6\rangle = 0, \tag{112}$$

$$L_{1new}|S_5\rangle - L_2'|S_3\rangle + L_{1new}^+|S_6\rangle = 0 \tag{113}$$

[4] The Lagrangian is defined, as usual, up to an overall factor

and the gauge transformations

$$\delta|S_1\rangle = L^+_{1new}|\lambda_1\rangle + L^+_{2new}|\lambda_2\rangle, \qquad \delta|S_4\rangle = L'_2|\lambda_1\rangle + L^+_{2new}|\lambda_4\rangle, \tag{114}$$
$$\delta|S_2\rangle = L_{1new}|\lambda_1\rangle - |\lambda_2\rangle + L^+_{2new}|\lambda_3\rangle, \qquad \delta|S_5\rangle = L'_2|\lambda_2\rangle + |\lambda_3\rangle - L^+_{1new}|\lambda_4\rangle, \tag{115}$$
$$\delta|S_3\rangle = L_{1new}|\lambda_2\rangle - L^+_{1new}|\lambda_3\rangle, \qquad \delta|S_6\rangle = -L'_2|\lambda_3\rangle + L_{1new}|\lambda_4\rangle, \tag{116}$$

where we denote

$$L'_2 = L_2 - \tfrac{1}{2}b_1^2 + b_2^+ b_2^2 - b_2\big(n + \tfrac{D-5}{2}\big). \tag{117}$$

With the help of these gauge transformations we get rid of the fields $|S_3\rangle$, $|S_4\rangle$, $|S_5\rangle$, $|S_6\rangle$ using parameters $|\lambda_2\rangle$, $|\lambda_1\rangle$ $|\lambda_3\rangle$, $|\lambda_4\rangle$ respectively. Then we make the field $|S_2\rangle = 0$ using the gauge parameter $|\lambda_1\rangle$. Now ones get the field $|S_1\rangle$ which obeys the equations of motion

$$L_0|S_1\rangle = L_{1new}|S_1\rangle = L'_2|S_1\rangle = 0 \tag{118}$$

and which has the dependence on the operators b_1^+, b_2^+. This dependence may be removed by the constrained gauge parameters $|\lambda_1\rangle$ and $|\lambda_2\rangle$. Finally we get the field

$$|S_1\rangle \;=\; a^{+\mu_1}\cdots a^{+\mu_n}\varphi_{\mu_1\ldots\mu_n}(x)|0\rangle \tag{119}$$

and no gauge transformation for them. Equations of motion (118) for the field (119) in component form are

$$(\partial^2 + m^2)\varphi_{\mu_1\ldots\mu_n}(x) = 0, \qquad \partial^{\mu_1}\varphi_{\mu_1\ldots\mu_n}(x) = 0, \qquad \varphi^{\nu}{}_{\nu\mu_3\ldots\mu_n}(x) = 0. \tag{120}$$

Thus we have shown that the Lagrangian (100) describes the massive bosonic higher spin field.

In Section 7 we explicitly construct Lagrangians for three fields with spin-1, spin-2, and spin-3 using our approach in terms of totally symmetric fields where all the fields and the gauge parameters have no off-shell constraints.

6 Unified description of all massive integer spin fields

In the previous section we considered the field with given spin and mass. Now we turn to consideration of fields with all integer spins together and find the Lagrangian describing the dynamics of such fields simultaneously.

It is evident, the fields with different spins $s = n$ may have the different masses which we denote m_n. First of all we introduce the state vectors with definite spin and mass as follows

$$|\chi, m\rangle_{n,m_n} \;=\; |\chi\rangle_n\,\delta_{m,m_n}, \tag{121}$$

with $|\chi\rangle_n$ being defined in (88) and m in (121) is now a new variable of the states $|\chi, m\rangle_{n,m_n}$. Second, we introduce the mass operator M acting on the variable m so that the states $|\chi, m\rangle_{n,m_n}$ are eigenvectors of the operator M with the eigenvalues m_n

$$M|\chi, m\rangle_{n,m_n} \;=\; m_n|\chi, m\rangle_{n,m_n} = m|\chi, m\rangle_{n,m_n}. \tag{122}$$

Construction of the Lagrangian decribing unified dynamics of fields with all spins is realized in terms of a single state $|\chi\rangle$ containing the fields of all spins (121)

$$|\chi\rangle \;=\; \sum_{n=0}^{\infty}|\chi,m\rangle_{n,m_n}. \tag{123}$$

It is naturally to assume that the Lagrangian describing a free dynamics of fields with all spins together should be a sum of all the Lagrangians for each spin (101)

$$\mathcal{L}=\sum_{n=0}^{\infty}\mathcal{L}_n=\sum_{n=0}^{\infty}\int d\eta_0\;{}_{n,m_n}\langle\chi|K_nQ_\sigma|\chi\rangle_{n,m_n}. \tag{124}$$

Here the operator Q_σ is defined by (92). This operator includes m^2 via L_0. L_{1new} and L^+_{1new}. Using the form of vectors $|\chi,m\rangle_{n,m_n}$ (121) and relation (122) we replace in (124) the operator Q_σ by the oprerator $Q_{\sigma M}$ which is obtained from the operator Q_σ (92) after substitution of the mass operator M instead of m.

Our aim is to rewrite (124) where any explicit dependence on n is absent. First, we rewrite the operator K_n in the form which is independent of n. This is done analogous to the case when we get Q_σ (92) from Q_n (91). It means, we should stand all $n+\frac{D-5}{2}$ to the right (or to the left) position and substitute σ instead of $n+\frac{D-5}{2}$. Let us denote this operator as K_σ.

Then we note that ${}_{n,m_n}\langle\chi|\chi\rangle_{n',m_{n'}}\sim\delta_{nn'}$ and due to $[Q_{\sigma M},\sigma]=0$ (82) we get

$${}_{n,m_n}\langle\chi|K_\sigma Q_{\sigma M}|\chi\rangle_{n',m_{n'}}\sim\delta_{nn'}. \tag{125}$$

Therefore eq. (124) may be transformed as

$$\begin{aligned}\mathcal{L} \;&=\; \sum_{n=0}^{\infty}\int d\eta_0\;{}_{n,m_n}\langle\chi|K_\sigma Q_{\sigma M}|\chi\rangle_{n,m_n}=\int d\eta_0\Big(\sum_{n=0}^{\infty}{}_{n,m_n}\langle\chi|\Big)K_\sigma Q_{\sigma M}\Big(\sum_{n'=0}^{\infty}|\chi\rangle_{n',m_{n'}}\Big)\\ &=\; \int d\eta_0\;\langle\chi|K_\sigma Q_{\sigma M}|\chi\rangle.\end{aligned} \tag{126}$$

The Lagrangian (126) describes a propagation of all integer spin fields with different masses simultaneously in terms of a single vector $|\chi\rangle$ containing fields of all spins.

Let us turn to the gauge transformations. Analogously to (121) we introduce the gauge parameters for the fields with given spin and mass

$$|\Lambda,m\rangle_{n,m_n}=|\Lambda\rangle_n\,\delta_{m,m_n}, \qquad |\Omega,m\rangle_{n,m_n}=|\Omega\rangle_n\,\delta_{m,m_n} \tag{127}$$

and analogously to (123) we denote

$$|\Lambda\rangle=\sum_{n=0}^{\infty}|\Lambda,m\rangle_{n,m_n}, \qquad |\Omega\rangle=\sum_{n=0}^{\infty}|\Omega,m\rangle_{n,m_n}, \tag{128}$$

with $|\Lambda\rangle_n$, $|\Omega\rangle_n$ being (102), (103) respectively. Summing up (104), (105) and (106) over all n we find gauge transformation for the field $|\chi\rangle$ (123) and transformation for the gauge parameter $|\Lambda\rangle$

$$\delta|\chi\rangle=Q_{\sigma M}|\Lambda\rangle, \qquad \delta|\Lambda\rangle=Q_{\sigma M}|\Omega\rangle. \tag{129}$$

Further we consider some examples following from the general construction developed in Sections 4, 5.

7 Examples

In order to elucidate the procedure of Lagrangian construction given in Section 5 we explicitly obtain the Lagrangians for fields with spin-1, spin-2, and spin-3 as examples. We will see that, in spite of all previous approaches, we actually get a description in terms of fields without any off-shell algebraic constraints.

7.1 Spin 1

Let us start with spin-1 field. In this case we have $n = 1$, $h = -\frac{D-3}{2}$ and taking into account the ghost numbers and the eigenvalues (88) of the fields (97), (107) and the gauge parameters (108) we write them as[5]

$$|S_1\rangle = \left[-ia^{+\mu}A_\mu(x) + b_1^+ A(x)\right]|0\rangle, \quad |A\rangle = \mathcal{P}_1^+ \varphi(x)|0\rangle, \quad |\lambda_1\rangle = \lambda(x)|0\rangle. \tag{130}$$

Substituting (130) into (100) and (104), (105) we get the Lagrangian[6]

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}A^\mu\Big[(\partial^2 + m^2)A_\mu - \partial_\mu\varphi\Big] + \frac{1}{2}A\Big[(\partial^2 + m^2)A - m\varphi\Big] \\ &\quad + \frac{1}{2}\varphi\Big[\varphi - \partial^\mu A_\mu - mA\Big] \end{aligned} \tag{131}$$

and the gauge transformations

$$\delta A_\mu = \partial_\mu\lambda, \qquad \delta A = m\lambda, \qquad \delta\varphi = (\partial^2 + m^2)\lambda. \tag{132}$$

Note that the gauge symmetry is Stückelberg.

We show that Lagrangian (131) is reduced to the Proca Lagrangian. First we note that field $\varphi(x)$ may be excluded from the Lagrangian with the help of its equation of motion. As a result we get

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2A^\mu A_\mu - mA\partial^\mu A_\mu - \frac{1}{2}\partial_\mu A\partial^\mu A, \tag{133}$$

where we denote $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Then we remove the Stückelberg field $A(x)$ with the help of its gauge transformation after that Lagrangian (133) is reduced to the standard Proca Lagrangian

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2A^\mu A_\mu. \tag{134}$$

[5]In order to have no imaginary unit i in the Lagrangians we will hereafter use $-ia^{+\mu}$ instead of $a^{+\mu}$ in the decompositions of the fields and the gauge parameters

[6]Since the Lagrangian (100) is defined up to an overall factor we multiply it by factor 1/2

7.2 Spin 2

Analogously to spin-1 case, we take into account the ghost numbers and the eigenvalues (88) of the fields (97), (107) and the gauge parameters (108) and write

$$
\begin{aligned}
|S_1\rangle &= \big\{\tfrac{(-i)^2}{2}a^{+\mu}a^{+\nu}h_{\mu\nu}(x) - ib_1^+a^{+\mu}h_\mu(x) + b_1^{+2}h_0(x) + b_2^+h_1(x)\big\}|0\rangle, && (135)\\
|S_2\rangle &= h_2(x)|0\rangle, && (136)\\
|A\rangle &= \mathcal{P}_1^+\big\{-ia^{+\mu}\varphi_\mu(x) + b_1^+\varphi(x)\big\}|0\rangle + \mathcal{P}_2^+\varphi_2(x)|0\rangle, && (137)\\
|\lambda_1\rangle &= \big\{-ia^{+\mu}\lambda_\mu(x) + b_1^+\lambda(x)\big\}|0\rangle, && (138)\\
|\lambda_2\rangle &= \lambda_2(x)|0\rangle. && (139)
\end{aligned}
$$

In the case under consideration, the relation (100) gives for the Lagrangian[7]

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{4}h^{\mu\nu}\Big\{(\partial^2+m^2)h_{\mu\nu} - 2\partial_\mu\varphi_\nu + \eta_{\mu\nu}\varphi_2\Big\}\\
& + \frac{1}{2}h^\mu\Big\{(\partial^2+m^2)h_\mu - m\varphi_\mu - \partial_\mu\varphi\Big\} - h_0\Big\{(\partial^2+m^2)h_0 - m\varphi + \tfrac{1}{2}\varphi_2\Big\}\\
& + \tfrac{D-1}{4}h_1\Big\{(\partial^2+m^2)h_1 - \varphi_2\Big\}\\
& + \frac{1}{2}h_2\Big\{(\partial^2+m^2)h_2 + \varphi_2 - \partial^\mu\varphi_\mu - m\varphi\Big\}\\
& + \frac{1}{2}\varphi^\mu\Big\{\varphi_\mu + \partial_\mu h_2 - \partial^\nu h_{\mu\nu} - mh_\mu\Big\} - \frac{1}{2}\varphi\Big\{\varphi + m(h_2 - 2h_0) - \partial^\mu h_\mu\Big\}\\
& - \frac{1}{2}\varphi_2\Big\{h_0 + \tfrac{D-1}{2}\,h_1 - h_2 + \tfrac{1}{2}\,h_\mu{}^\mu\Big\}. \qquad (140)
\end{aligned}
$$

The gauge transformations (104), (105) read

$$
\begin{aligned}
&\delta h_{\mu\nu} = \partial_\mu\lambda_\nu + \partial_\nu\lambda_\mu - \eta_{\mu\nu}\lambda_2, && \delta h_0 = m\lambda - \tfrac{1}{2}\lambda_2, && (141)\\
&\delta h_\mu = \partial_\mu\lambda + m\lambda_\mu, && \delta h_1 = \lambda_2, && (142)\\
&\delta\varphi_\mu = (\partial^2+m^2)\lambda_\mu, && \delta h_2 = \partial^\mu\lambda_\mu + m\lambda - \lambda_2, && (143)\\
&\delta\varphi = (\partial^2+m^2)\lambda, && \delta\varphi_2 = (\partial^2+m^2)\lambda_2. && (144)
\end{aligned}
$$

Here we see again that the gauge symmetry is Stückelberg.

Let us show that Lagrangian (140) is reduced to the Fierz-Pauli Lagrangian. Let us first get rid of the fields h_μ, h_1, h_0 using their gauge transformations and then remove fields φ_μ, φ using their equations of motion. Ones obtain

$$
\begin{aligned}
\mathcal{L} \;=\; & \frac{1}{4}\partial^\sigma h^{\mu\nu}\partial_\sigma h_{\mu\nu} - \frac{1}{2}\partial^\sigma h_{\sigma\mu}\partial_\nu h^{\mu\nu} - \frac{1}{4}m^2h^{\mu\nu}h_{\mu\nu}\\
& - h_2\partial^\mu\partial^\nu h_{\mu\nu} - \partial_\mu h_2\partial^\mu h_2 + m^2h_2h_2 + \varphi_2\big(h_2 - \tfrac{1}{2}h_\mu{}^\mu\big). \qquad (145)
\end{aligned}
$$

Then we use equation of motion $h_2 = \frac{1}{2}h_\mu{}^\mu$ and arrive at the Fierz-Pauli Lagrangian

$$
\begin{aligned}
\mathcal{L}_{FP} \;=\; & \frac{1}{4}\partial^\sigma h^{\mu\nu}\partial_\sigma h_{\mu\nu} - \frac{1}{4}\partial_\sigma h^\mu{}_\mu\partial^\sigma h^\nu{}_\nu - \frac{1}{2}\partial^\sigma h_{\sigma\mu}\partial_\nu h^{\mu\nu}\\
& - \frac{1}{2}h^\sigma{}_\sigma\partial^\mu\partial^\nu h_{\mu\nu} - \frac{1}{4}m^2h^{\mu\nu}h_{\mu\nu} + \frac{1}{4}m^2h^\mu{}_\mu h^\nu{}_\nu. \qquad (146)
\end{aligned}
$$

[7]Lagrangians (140) is Lagrangian (100) multiplied by $-1/2$

7.3 Spin 3

Taking into account the ghost numbers and the eigenvalues (88) we write the fields (97), (107)

$$
\begin{aligned}
|S_1\rangle &= \big\{\tfrac{(-i)^3}{3!}\,a^{+\mu}a^{+\nu}a^{+\sigma}h_{\mu\nu\sigma}(x)+\tfrac{(-i)^2}{2}\,a^{+\mu}a^{+\nu}b_1^+h_{\mu\nu}(x)-ia^{+\mu}b_1^{+2}h_\mu(x) \\
&\qquad +b_1^{+3}h_0(x)-ib_2^+a^{+\mu}h_{1\mu}(x)+b_2^+b_1^+h_1(x)\big\}|0\rangle, && (147)\\
|S_2\rangle &= \big\{-ia^{+\mu}h_{2\mu}(x)+b_1^+h_2(x)\big\}|0\rangle, && (148)\\
|S_3\rangle &= h_3(x)|0\rangle, \qquad\qquad |S_4\rangle = h_4(x)|0\rangle, && (149)\\
|A\rangle &= \mathcal{P}_1^+\big\{\tfrac{(-i)^2}{2}\,a^{+\mu}a^{+\nu}\varphi_{\mu\nu}(x)-ia^{+\mu}b_1^+\varphi_\mu(x)+b_1^{+2}\varphi_0(x)+b_2^+\varphi(x)\big\}|0\rangle \\
&\quad +\mathcal{P}_2^+\big\{-ia^{+\mu}\varphi_{2\mu}(x)+b_1^+\varphi_2(x)\big\}|0\rangle, && (150)
\end{aligned}
$$

the gauge parameters (108)

$$
\begin{aligned}
|\Lambda_0\rangle &= \mathcal{P}_1^+\big\{\tfrac{(-i)^2}{2}\,a^{+\mu}a^{+\nu}\lambda_{\mu\nu}(x)-ia^{+\mu}b_1^+\lambda_\mu(x)+b_1^{+2}\lambda_0(x)+b_2^+\lambda(x)\big\}|0\rangle \\
&\quad +\mathcal{P}_2^+\big\{-ia^{+\mu}\lambda_{2\mu}(x)+b_1^+\lambda_2(x)\big\}|0\rangle, && (151)\\
|\Lambda_1\rangle &= \mathcal{P}_1^+\mathcal{P}_2^+\lambda_5(x)|0\rangle, && (152)
\end{aligned}
$$

and the parameters (106) for symmetry transformations of the gauge parameters

$$
|\Omega\rangle = \mathcal{P}_1^+\mathcal{P}_2^+\omega(x)|0\rangle. \qquad (153)
$$

Then we get Lagrangian

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{6}\,h^{\mu_1\mu_2\mu_3}\Big\{(\partial^2+m^2)h_{\mu_1\mu_2\mu_3}-3\partial_{\mu_1}\varphi_{\mu_2\mu_3}+3\eta_{\mu_1\mu_2}\varphi_{2\mu_3}\Big\}\\
&\quad +\frac{1}{2}\,h^{\mu\nu}\Big\{(\partial^2+m^2)h_{\mu\nu}-m\varphi_{\mu\nu}-2\partial_\mu\varphi_\nu+\eta_{\mu\nu}\varphi_2\Big\}\\
&\quad -2h^\mu\Big\{(\partial^2+m^2)h_\mu-\partial_\mu\varphi_0-m\varphi_\mu+\tfrac{1}{2}\varphi_{2\mu}\Big\}\\
&\quad +\tfrac{D+1}{2}h_1^\mu\Big\{(\partial^2+m^2)h_{1\mu}-\partial_\mu\varphi-\varphi_{2\mu}\Big\}\\
&\quad +6h_0\Big\{(\partial^2+m^2)h_0-m\varphi_0+\tfrac{1}{2}\varphi_2\Big\}-\tfrac{D+1}{2}h_1\Big\{(\partial^2+m^2)h_1-m\varphi-\varphi_2\Big\}\\
&\quad +h_2^\mu\Big\{(\partial^2+m^2)h_{2\mu}-\partial^\nu\varphi_{\mu\nu}-m\varphi_\mu+\varphi_{2\mu}\Big\}\\
&\quad -h_2\Big\{(\partial^2+m^2)h_2-\partial^\mu\varphi_\mu-2m\varphi_0+\varphi_2\Big\}\\
&\quad -h_4\Big\{(\partial^2+m^2)h_3-\partial^\mu\varphi_{2\mu}-m\varphi_2\Big\}-h_3\Big\{(\partial^2+m^2)h_4+\tfrac{1}{2}\varphi^\mu{}_\mu+\varphi_0+\tfrac{D+1}{2}\varphi\Big\}\\
&\quad +\frac{1}{2}\varphi^{\mu\nu}\Big\{\varphi_{\mu\nu}-\partial^\sigma h_{\sigma\mu\nu}-mh_{\mu\nu}+2\partial_\mu h_{2\nu}-\eta_{\mu\nu}h_3\Big\}\\
&\quad -\varphi^\mu\Big\{\varphi_\mu-\partial^\nu h_{\mu\nu}+\partial_\mu h_2+m(h_{2\mu}-2h_\mu)\Big\}\\
&\quad +2\varphi_0\Big\{\varphi_0-\partial^\mu h_\mu+m(h_2-3h_0)-\tfrac{1}{2}h_3\Big\}-\tfrac{D+1}{2}\varphi\Big\{\varphi-\partial^\mu h_{1\mu}-mh_1+h_3\Big\}\\
&\quad -\varphi_2^\mu\Big\{\tfrac{1}{2}h_{\mu\nu}{}^\nu+h_\mu+\tfrac{D+1}{2}h_{1\mu}-h_{2\mu}+\partial_\mu h_4\Big\}\\
&\quad +\varphi_2\Big\{\tfrac{1}{2}h_\mu{}^\mu+3h_0+\tfrac{D+1}{2}h_1-h_2+mh_4\Big\}, && (154)
\end{aligned}
$$

the gauge transformations for the fields

$$
\begin{aligned}
\delta h_{\mu_1\mu_2\mu_3} &= \partial_{\mu_1}\lambda_{\mu_2\mu_3} + \partial_{\mu_2}\lambda_{\mu_3\mu_1} + \partial_{\mu_3}\lambda_{\mu_1\mu_2} \\
&\qquad - \eta_{\mu_1\mu_2}\lambda_{2\mu_3} - \eta_{\mu_2\mu_3}\lambda_{2\mu_1} - \eta_{\mu_3\mu_1}\lambda_{2\mu_2}, && (155)\\
\delta h_{\mu\nu} &= m\lambda_{\mu\nu} + \partial_\mu\lambda_\nu + \partial_\nu\lambda_\mu - \eta_{\mu\nu}\lambda_2, && (156)\\
\delta h_\mu &= m\lambda_\mu + \partial_\mu\lambda_0 - \tfrac{1}{2}\lambda_{2\mu}, \qquad \delta h_0 = m\lambda_0 - \tfrac{1}{2}\lambda_2, && (157)\\
\delta h_{1\mu} &= \partial_\mu\lambda + \lambda_{2\mu}, \qquad \delta h_1 = m\lambda + \lambda_2, && (158)\\
\delta h_{2\mu} &= \partial^\nu\lambda_{\mu\nu} + m\lambda_\mu - \lambda_{2\mu},, \qquad \delta h_2 = \partial^\mu\lambda_\mu + 2m\lambda_0 - \lambda_2, && (159)\\
\delta h_3 &= \partial^\mu\lambda_{2\mu} + m\lambda_2 - \lambda_5, \qquad \delta h_4 = -\tfrac{1}{2}\lambda^\mu{}_\mu - \lambda_0 - \tfrac{D+1}{2}\lambda, && (160)\\
\delta\varphi_{\mu\nu} &= (\partial^2 + m^2)\lambda_{\mu\nu} - \eta_{\mu\nu}\lambda_5, \qquad \delta\varphi_\mu = (\partial^2 + m^2)\lambda_\mu, && (161)\\
\delta\varphi_0 &= (\partial^2 + m^2)\lambda_0 - \tfrac{1}{2}\lambda_5, \qquad \delta\varphi = (\partial^2 + m^2)\lambda + \lambda_5, && (162)\\
\delta\varphi_{2\mu} &= (\partial^2 + m^2)\lambda_{2\mu} - \partial_\mu\lambda_5, \qquad \delta\varphi_2 = (\partial^2 + m^2)\lambda_2 - m\lambda_5, && (163)
\end{aligned}
$$

and the gauge transformation for the gauge parameters

$$
\begin{aligned}
&\delta\lambda_{\mu\nu} = \eta_{\mu\nu}\omega, \qquad \delta\lambda_\mu = 0, \qquad \delta\lambda_0 = \tfrac{1}{2}\omega, && (164)\\
&\delta\lambda = -\omega, \qquad \delta\lambda_{2\mu} = \partial_\mu\omega, \qquad \delta\lambda_2 = m\omega, && (165)\\
&\delta\lambda_5 = (\partial^2 + m^2)\omega. && (166)
\end{aligned}
$$

Here we see that the gauge symmetry is again Stückelberg.

8 Summary

We have developed the BRST approach to derivation of gauge invariant Lagrangians for bosonic massive higher spin gauge fields in arbitrary dimensional Minkowski space. We studied the closed algebra of the operators generated by the constraints which are necessary to define an irreducible massive integer spin representation of Poincare group and constructed new representation for this algebra. It is shown that the BRST operator corresponding to the algebra with new expressions for the operators generates the correct Lagrangian dynamics for bosonic massive fields of any value of spin. We construct Lagrangians in the concise form both for the field of any given spin and for fields of all spins propagating simultaneously in arbitrary space-time dimension. As an example of general scheme we obtained the Lagrangian and the gauge transformations for the spin-1, spin-2, and spin-3 massive fields in the explicit form without any gauge fixing.

The main results of the paper are given by the relations (100), (101) where Lagrangian for the massive field with arbitrary integer spin is constructed, and (104)–(106) where the gauge transformations for the fields and the gauge transformations the gauge parameters are written down. In the case of Lagrangian describing propagation of all massive bosonic fields simultaneously the corresponding relations are given by the formulas (126), (129) for the Lagrangian and the gauge transformations respectively.

The procedure for Lagrangian construction developed here for higher spin massive bosonic fields can also be applied to bosonic and fermionic higher spin theories in AdS background. There are several possibilities for extending our approach. This approach

may be applied to Lagrangian construction for massive and massless mixed symmetry tensor or tensor-spinor fields (see [15] for corresponding bosonic massless case), for Lagrangian construction for fermionic massive fields and for supersymmetric higher spin models.

It is interesting to note that the same result for Lagrangian construction (100) of massive bosonic higher spin fields could be obtained if we start with the massless bosonic operator algebra. Unlike [18] where the 'minimal' set of operators was modified it is possible to construct more general representation of the algebra [20] which has two arbitrary parameters. First of these parameters is the same as in the case of 'minimal' modification of the algebra and defines spin of the field, the other parameter has dimension of mass squared and is identified with the mass of the field.

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